EngOpt 2012 – 3rd International Conference on Engineering Optimization
Rio de Janeiro, Brazil, 01 - 05 July 2012.

Automatic History Matching Considering Surrogate Based Optimization and Karhunen-Loève Expansions

José Dásio de Lira Jr [1,2], Silvana M. B. Afonso [1], Ramiro B. Willmersdorf [1], Bernardo Horowitz [1]

[1] Federal University of Pernambuco, Civil Engineering Department, Recife – PE – Brazil
[2] Federal Institute of Education, Science and Technology of Pernambuco, DACI/IFPE, Recife – PE – Brazil
josedasio@recife.ifpe.edu.br, smb@ufpe.br, ramiro@willmersdorf.net, horowitz@ufpe.br

1. Abstract
In oil reservoir engineering one stage of great interest is the production history matching process, by which the numerical model is adjusted to reproduce the observed field production. The objective is to find the model parameters which minimize the difference between calculated and observed fluid production rates. However, the consideration of the rock properties along every cell of the numerical model that represents the reservoir as design variables would lead to an unfeasible large number of variables and consequently, a very difficult problem to be solved. To overcome that, we will assume that the permeability field will be stochastic and represented by a spectral decomposition through Karhunen-Loève expansion (KLE)**.** In this work both linear and nonlinear forms for KLE expansions will be investigated. The mathematical formulation of this problems leads to nonsmooth, and high cost functions leading to a difficult problem to be tackled. To overcome that, surrogate models are built and incorporated in a sequential optimization strategy. Results are presented for two different reservoirs. The potential of the developed tool is discussed.

2. Keywords: **Petroleum Engineering, Optimization, Reservoir Simulation, Uncertainty, History Matching**

3. Introduction
The history matching (HM) problem aims to validate and improve the characterization and fluid flow numerical models for an oil field. It is a problem of inversion of production data in order to obtain reliable estimates of physical properties of the reservoir (e.g. porosity, permeability). [1].
The most common used method for history matching is to execute many simulations changing a few parameters at a time in a trial-and-error fashion. This is a very costly and time consuming approach as in general it requires months of effort and many simulations to achieve a good solution, and often may not provide a good predictor of future field performance. Automatic HM can be conducted by optimization techniques, which is in general a more effective way to solve this problem. Given a set of observed data one seeks the physical properties (design variables) values from which, in a general sense, the difference between the observed and calculated response is minimized.
In this work, the permeability of the rock reservoir will be the quantity to be adjusted. However, the consideration of the rock properties along every cell of the numerical model that represents the reservoir as design variables would lead to an unfeasible large number of variables and consequently, a very difficult problem to be solved. To overcome that, we will assume that the permeability field will be stochastic and represented by a spectral decomposition through Karhunen-Loève expansion (KLE). In this work both linear and nonlinear forms for KLE expansions [2] will be used.
The Karhunen-Loeve Expansion (KLE) is a mathematical technique to represent a stochastic field (the permeability, for instance) with a know covariance matrix, where the stochastic field is represented as a series expansion. The terms of this series are products of an eigenvalue and its corresponding eigenvector of the covariance matrix and a random standard normal variable. The summation of this series for all eigenvalues of the covariance matrix with specific samples taken from standard normal distribution provides a realization of the field that preserves statistical moments (up to the second one, higher ones are possible with more sophisticated implementations). The main idea of the KLE is to have a parametrization of the random input field in terms of independent standard normal variables, and this parametrization can be used in optimization algorithms. The covariance matrix of the input field can be easily computed from realizations of the field of interest the come from a geostatiscal analysis.
One important difficulty of the use of the KLE in reservoir problems is that the covariance matrix can be very large, since it is a square matrix with number of rows and columns equal to the number of cells in the domain. Computing the eigenpairs of this matrix, for a large reservoir model, can be a daunting task. It may be unnecessary to compute all the eigenpairs, since the series can be truncated after some terms and still represent well the statistical properties of the input field. As the KLE requires high computational resources, kernel Karhunen-Loève expansion (KKLE) is used to minimize this drawback. The KLE will be used in this work, because not only it computes the eigenpairs of the kernel matrix, which is smaller than the full covariance matrix, but also it can be extended without serious difficult to preserve higher order moments of the original fields, if so desired, and this can represent geological structures such facies and channels that are lost when only two points statistics are preserved.
Apart from that, HM by optimization techniques commonly involves several calls of the numerical simulator, which may turn the optimization task into a very time consuming process. In order to ameliorate such drawback, surrogate models using Kriging based data fitting [3, 4] will be employed in substitution to the numerical reservoir simulations. In this sense, to tackle the HM problems, here optimization tools will be used in which the optimization algorithm will operate solely on the substitute model to be constructed based on simulation results obtained for a limited number of designs generated by a design of experiments technique (DOE) [5]. The main characteristics of Kriging strategy are: to easily accommodate irregular distributed sample data, and the ability to model multimodal functions with many peaks and valleys, such the ones obtained for the applications addressed here.

From a proper choice of a DOE scheme [5], followed by the evaluations of the true (high fidelity) function at the samplings, a Kriging predictor is built in order to evaluate the functions at untried points during the optimization algorithm iterations. The local optimization algorithm of choice is the sequential quadratic programming (SQP) [6]. This will be embedded here in an interactive procedure, named sequential approximate optimization (SAO) [7]. A trust region based method [7] is used to update the design variable space for each local (subproblem) optimization solution (SAO iteration). The required numerical reservoir simulations to build the Kriging models are performed using IMEX commercial software from [8].
The case study presented here is a waterflooding problem in oil reservoir with one injection and two production wells [9]. The application illustrates the use of the presented methodology to conduct the HM of the reservoirs from which synthetic permeability fields created from a given analytic covariance function are used. The results obtained show that our method is able to reconstruct the unknown permeability distribution in a reliable and efficient way from synthetic data provided by a numerical solution.

4. Representation of The Permeability Field

In this work, the permeability of the rock reservoir will be the quantity to be adjusted. However, adopting the rock properties along every cell of the numerical model that represent the reservoir as design variable leads to a very difficult problem to be solved as a large number of design variables appears, when one tries to characterize a real life reservoir. To overcome that we will assume that the permeability field $\boldsymbol{y}$ will be stochastic and represented by a spectral decomposition through Karhunen-Loève expansion [10, 11, 2]. In this work both linear and nonlinear forms for KLE expansion will be investigated.

4.1. Linear KL

Supposing there is a set of centered realizations of a random field, given by $y_k, k = 1..., N_r$, where N_r is the number of realizations and each vector has N_c elements, corresponding in the case of reservoir simulations to the value of the property of interest in the cell of the reservoir. The covariance matrix of the input field can be computed with

$$\mathbf{C} = \frac{1}{N_r} \sum_{j=1}^{Nr} y_j y_j^T \tag{1}$$

where it is expected that the number of realizations is large enough to ensure convergence of the matrix. Through the KLE, new realizations of the field $\boldsymbol{y}$ can be computed in terms of the eigenvalues and eigenvectors of the covariance matrix, as in:

$$\boldsymbol{y} = \mathbf{E}\boldsymbol{\Lambda}^{1/2}\boldsymbol{\xi} \tag{2}$$

In which $\boldsymbol{\xi}$ is a vector of random standard normal variables and $\mathbf{E}$, $\boldsymbol{\Lambda}$ are respectively, the eigenvector and the diagonal eigenvalue matrix of the covariance matrix given in Eq. (6). According to Scholkopf et al [19] to obtain the KLE the following eigvector/eigvalue equation needs to be solved.

$$\lambda v = Cv \tag{3}$$

The solution of Eq. (3) could be very costly as the dimension of C is N_c x N_c, where N_c is the number of cells in the model. This problem can be overcome applying a kernel technique [2] in which the eigenvector and eigenvalues presented in Eq. (2) are obtained from the solution of a reduced order (N_r x N_r) eigenvalue/eigenvector problem. The procedure is briefly described below. Pre and post multiplying Eq. (1) by v, leads to:

$$Cv = \frac{1}{N_r} \sum_{j=1}^{N_r} \left(y_j . v\right) y_j \tag{4}$$

And considering Scholkopf et al [19] assumptions in which the eigenvectors can be written as linear combination of provided realizations:

$$v = \sum_{j=1}^{N_r} \alpha_j y_j \tag{5}$$

Combining Equations 3, 4 and 5 leads the following form:

$$\lambda \sum_{i=1}^{N_r} \alpha_i \left(y_k . y_i\right) = \frac{1}{N_r} \sum_{i=1}^{N_r} \alpha_i \left(y_k . \sum_{j=1}^{N_r} y_j \right) \left(y_j . y_i\right) \tag{6}$$

Defining $\boldsymbol{K}$ as the kernel matrix (dimension N_r x N_r), from which each element is obtained as

$$K_{ij} = \left(y_i . y_j \right) \tag{7}$$

Where y_i , y_j are realizations belonging to set $y = \{y_1, y_2, \ldots, y_{Nr}\}$. Through the above definition, Eq. (6) can be written in a compact form such as

$$N_r \lambda K \alpha = K^2 \alpha \tag{8}$$

Which leads to:

$$N_r \lambda \alpha = K \alpha \tag{9}$$

The above equation is known as the kernel eigenvector/eigenvalue equation [19]. Solving such equation the solution of Eq. (3) will also be obtained using the relation of Eq. (5). Moreover Eq. (9) involves a problem of dimension N_r x N_r while the original one Eq. (3) has dimension N_c x N_c , as in general $N_r << N_c$ the solution of Eq. (3) is obtained in a much more efficient way.

4.2. Nonlinear KL

In this case, the KL expansion is developed in a space F of order greater than the original space (R^{Nc}) of realizations, where space F is not linearly related to the original space R^{Nc} . This characteristic setting the nonlinear KL expansion, also called nonlinear KKLE when a kernel technique is used [2]. The basic idea of nonlinear KL expansion can be seen in Figure 1, where realizations are in R^2.

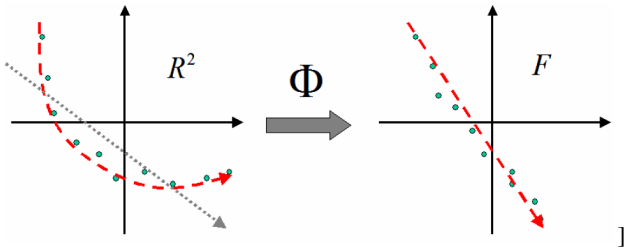

Figure 1. The basic idea of nonlinear KL [2].

As noted in Figure 1, nonlinear KL expansion is performed in high-dimensional space F. The concepts developed in the linear KL expansion to solve the kernel linear eigenvalue problem Eq. (9) are the same, only now applied in high-order space F.
Considering a nonlinear mapping Φ which relates the original design space R^{Nc} to another space F:

$$\Phi : R^{N_c} \rightarrow F; \quad Y = \Phi(y); \qquad y \in R^{N_c}, Y \in F. \tag{10}$$

Now the covariance matrix of the space F is given by:

$$W = \frac{1}{N_r} \sum_{j=1}^{N_r} \Phi(y_j) \Phi(y_j)^T \tag{11}$$

The size of the covariance matrix W is N_F x N_F , where $N_F > N_c$. Similarly as was developed for the linear KL expansion:

$$\lambda v = W v \tag{12}$$

Where v is a eigenvector of W and λ is a eigenvalue of W. Instead of solving this equation directly, the eigenvalue problem is formulated as shown below.

$$N_r \lambda \alpha = K \alpha \tag{13}$$

In this case, K is defined as follow.

$$K : K_{ij} = \left(\Phi(y_i) . \Phi(y_j) \right) \tag{14}$$

Now each element of the kernel matrix is an inner product of vectors in the space F. The dimension of the kernel matrix remains N_r x N_r, as in the linear KKLE. The inner product calculation can be performed efficiently with the kernel function given by:

$$(\Phi(x).\Phi(y)) = k(x, y) \tag{15}$$

The kernel function $k(x,y)$ calculates the inner product space F directly from the elements in the original space R^{Nc}. The kernel polynomial function used in this work is defined as:

$$(\Phi(x).\Phi(y)) = k(x, y) = (x.y)^d \tag{16}$$

Where d is the order of the polynomial function. Developing the KL expansion in space F with polynomial of order d corresponds to preserve the moments of order $2d$. In order to preserve not only the moments of order $2d$, but also all other moments of the lowest order, the following kernel function is used:

$$(\Phi(x).\Phi(y)) = k(x, y) = \sum_{i=1}^{d} (x.y)^i \tag{17}$$

After expansion of KL in the space F, the realization will be produced in this space. The realizations need to be at original space R^{Nc}, so it is necessary to develop an inverse mapping such as:

$$y = \Phi^{-1}(Y) \tag{18}$$

The pre-image technique is used to perform the inverse mapping [2, 19].

5. Problem Formulation

HM problem [1] aims to validate and improve the characterization and fluid flow numerical models for an oil field. It is a problem of inversion of production data in order to obtain reliable estimates of physical properties of the reservoir (e.g. porosity, permeability).
In this sense, given an observed data one much seek the physical properties (design variables) values from which, in the general sense, the difference between the observed and calculated response is to be minimized.
Mathematically, the HM problem is formulated as

$$\begin{aligned} &\text{Minimize} \quad f_{error}\left(y(\xi)\right) \\ &\text{Subject to:} \quad \xi_i^L \le \xi_i \le \xi_i^U \quad i = 1, n_{dv} \end{aligned} \tag{19}$$

In which f_{error} is a function that quantifies the difference between the results obtained when considering the two models (observed and calculated), $\boldsymbol{y}$ is the permeability field defined represented by K-L expansion, and n_{dv} is the total number of design variables and ξ_i^L, ξ_i^U , are respectively, it's lower and upper bounds values (that could be stochastic or not) .

In the present work the output response considered is the cumulative oil distribution and the metric consider for f_{error} is

$$f_{error}\left(y(\xi)\right) = \sum_{n=1}^{t} \sum_{i=1}^{N_p} \left\{ f_i\left(y(\xi)\right)^n - d_{obs_i}^n \right\}^2 \tag{20}$$

Where N_p is the number of producers, f_i is the calculated cumulative oil production at n^{th} time step, and $d_{obs_i}^n$ is the observed cumulative oil production at n^{th} time step and t is the total number of time steps.

6. Surrogate Models

Surrogate models are built to provide smooth functions with sufficient accuracy for fast evaluations for both uncertainty quantification and in the optimization process.
Kriging [3, 4] methodology is considered in this work. The central idea of this scheme is to assume that errors in the model are not independent but rather exhibit spatial correlation related to the distance between corresponding points modeled by a Gaussian process around each sample point.
The main advantages of this scheme are easily to accommodate irregularly distributed sample data, and the ability to model multimodal functions with many peaks and valleys. Moreover, kriging models provide exact interpolation at the sample points.
Kriging is a data fitting based model, the first step is to generate the sampling points. These points in the design space are determined by DOE [5] approach. In this work Latin Centroidal Voronoi Telsselations (LCVT) is used [12]. Next, the kriging fitting scheme is used to develop the predictor and error expressions in order to evaluate the functions at untried design points. Details of such procedure can be found elsewhere [4, 5, 9, 13, 14].

7. Optimization Strategy

To tackle the problem formulated in section 2 the SAO methodology [3, 7, 14] is employed. In this procedure, surrogate smooth functions are used in place of the computationally expensive and/or nonsmooth functions in a sequence of optimization steps, which are confined to a small region of the parameter space. To update the design space of each optimization subproblem a trust region based scheme [7, 15] is used.

Mathematically, each subproblem *k* is defined as:

$$\begin{aligned} &\text{Maximize } \hat{f}^k(\boldsymbol{x}) \\ &\text{Subject to: } \hat{g}_i^k(\boldsymbol{x}) \le 0, \;\; i = 1, \ldots, m \\ &\qquad \boldsymbol{x}_l \le \boldsymbol{x}_l^k \le \boldsymbol{x} \le \boldsymbol{x}_u^k \le \boldsymbol{x}_u, \quad k = 0, 1, 2, \ldots, k_{\max} \\ &\text{Where: } \quad \boldsymbol{x}_l^k = \boldsymbol{x}_c^k - \Delta^k \\ &\qquad \boldsymbol{x}_u^k = \boldsymbol{x}_c^k + \Delta^k \end{aligned} \tag{21}$$

In above equations, $\hat{f}^k(\boldsymbol{x})$ and $\hat{g}^k(\boldsymbol{x})$ are respectively the approximated (surrogate) objective and constraints functions, $\boldsymbol{x}_c^k$ is the center point of the trust region, Δ^k is the width of the trust region and $\boldsymbol{x}_l^k$, $\boldsymbol{x}_u^k$ are respectively the lower and upper bounds of the design variables for SAO iteration *k*.

7.1 SAO Algorithm

Each subproblem described above defines a SAO iteration. To update the trust region size Δ^k for each optimization subproblem we considered the approach described in Alexandrov et al [7] which takes into consideration the accuracy of surrogate functions against the true functions. The main steps involved in the computations are:

1. Compute the expensive and/or nonsmooth objective function and constraints at the central point in the subregion.
2. Construct surrogate model in the subregion.
3. Optimize within the subregion using the surrogate objective function and constraints.
4. Compute the true objective function and constraints at the optimum identified in Step 3.
5. Check for convergence.
6. Move/shrink/expand the subregion according to the accuracy of the approximated model compared to the true function and constraint values.
7. Impose C^0 local consistency.
8. Check for overall optimization convergence. If it is achieved stop the SAO procedure; otherwise return to Step 3.

8. Example

In this section the tools described in this paper are employed for a study case considering both linear and nonlinear KL expansion to represent the permeability field. The studies are conducted using DAKOTA platform [17]. The SAO runs use SQP/NPSOL as the optimizer of choice.

8.1 Problem Definition

The investigated example is the one indicated in Figure 2. It has three wells: two producers and one injector. The injector rate is fixed at 24 m^3/day while the sum of the rates of the producers is fixed at 24 m^3/day with a prescribed minimum button hole pressure (BHP) of 50 Kgf/cm^2. The reservoir has a volume of 510 x 510 x 4 m^3 which is discretized considering a 51 x 51 x 1 grid. The production time considered used for the HM is ten years and a further extrapolation of 6 years is considered for future prediction. The realizations were generated considering the following covariance function [18]:

$$\text{cov} = \sigma_E e^{-\frac{(x_1 - x_2)}{b_1} - \frac{(y_1 - y_2)}{b_2}} \tag{22}$$

In the above equation (x_1,y_1) , (x_2,y_2) are the coordinates of two generic points on the grid, b_1 e b_2 are correlation length and σ_E is the standard deviation. The following values are attributed for the studies: b_1= 6 , b_2= 9 and σ_E = 1. A Part from that, in the process of generation of the realizations it was considered two regions with different permeability mean values. A total of 1000 realizations were generated, from which one of them is chosen as the observed case. Figure 3 illustrates some possible realizations for the studied case. The selected permeability field (observed case) is illustrated in Figure 4.

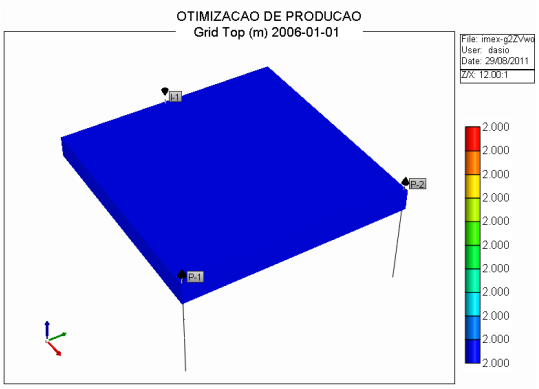

Figure 2. Reservoir model.

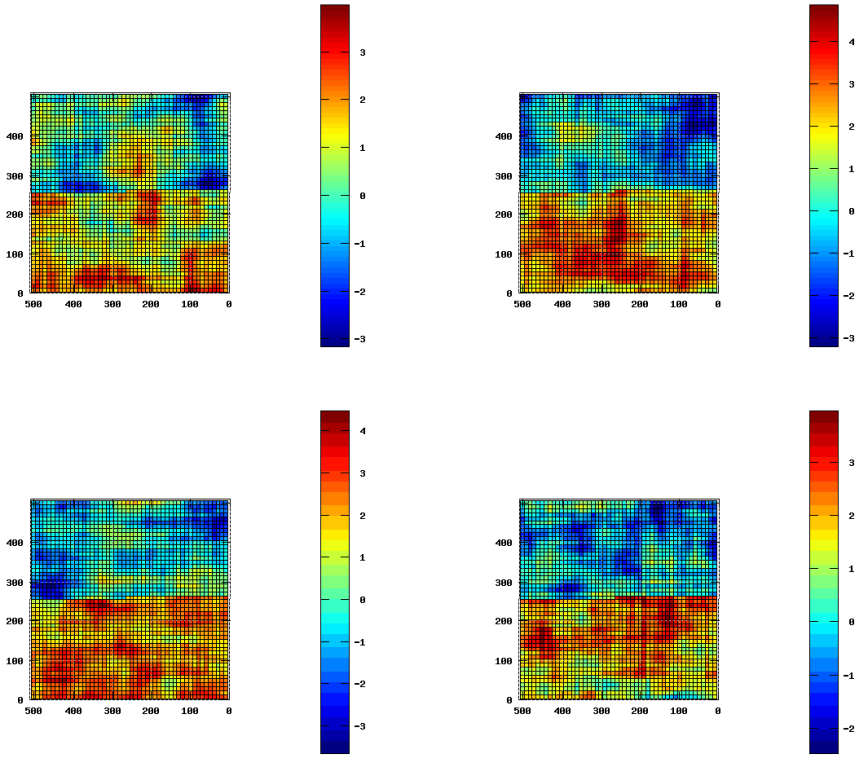

Figure 3. Possible realizations for the case study (logarithmic scale).

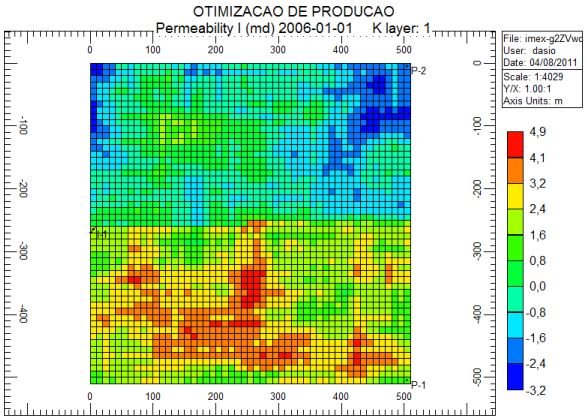

Figure 4. Realization chosen as observed case.

To conduct optimization both linear and nonlinear KKLE (Kernel Karhunen-Loève expansion) with ten principal component analysis is considered to represent the permeability field. The initial design variables is (ξ_1 = 0,5 and ξ_i = 0, i = 2,.., 10). The lower and upper bound values are respectively (ξ^L_i = -8 , i = 1,…,10), (ξ^U_i = 8 , i = 1,…,10).

8.2 HM with linear KKLE

For this specific case the initial permeability field is indicated in Figure 5. The normalized initial value for f_{error} is 0.5276. The optimization results are show in Table 1.

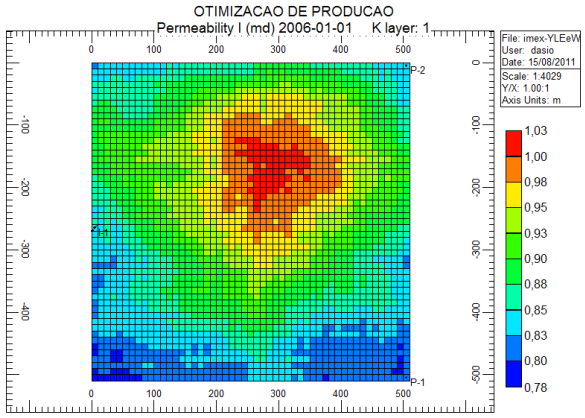

Figure 5. Initial estimate of the permeability field.

Table 1. Optimum Values, HM with linear KKLE.

ξ_1	ξ_2	ξ_3	ξ_4	ξ_5	ξ_6	ξ_7	ξ_8	ξ_9	ξ_{10}	f_{error}
0.5430	2.9719	2.2632	1.5829	3.0909	-0.5986	3.3174	-1.4232	2.4135	3.5417	0.0015

As can be observed, at the end of the optimization process the objective function has decrease substantially. The optimum converge in 87 SAO iterations.

The cumulative oil production obtained with the permeability field using the ten optimum principal components of the Table 1 are plotted (red) in Figure 6. For comparison purposes the initial (green) and observed (blue) cumulative oil production are also plotted in this figure. As can be seen both matching and future prediction were satisfactory. However the calculated permeability field (Figure 7) is very different from the one taken as real (Figure 4). This is mainly observed in the central region of the field and it happened due to the limitation of the linear KKLE. In order to overcome that and with the aim to obtain a field with closer characteristics of the observed field the nonlinear KKLE is considered next to solve this problem.

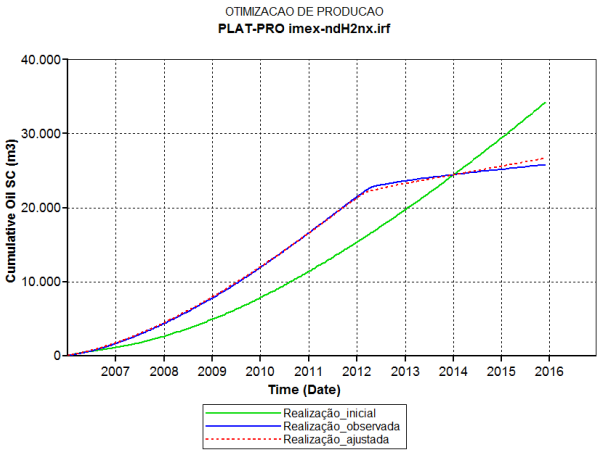

Figure 6. Curves of cumulative oil production obtained in the history matching process.

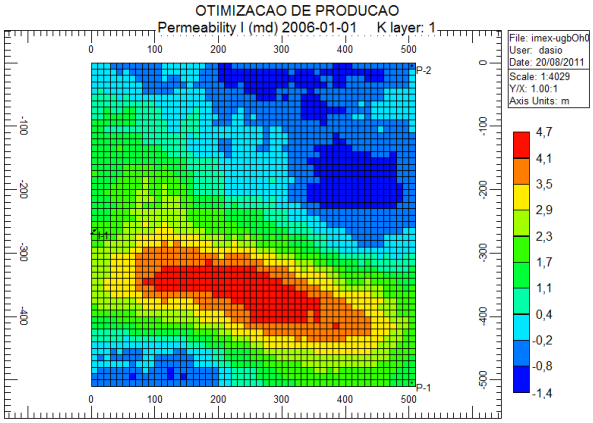

Figure 7. Permeability Field obtained in the history matching process.

8.3 HM with nonlinear KKLE

When considering the nonlinear KKLE the initial permeability field considered is the one indicated in Figure 8. A much faster convergence was observed for this case, tacking only 12 SAO iterations. The normalized initial value for f_{error} is 2.8638. The optimum results are those indicated in Table 2.

Table 2. Optimum Values, HM with nonlinear KKLE.

ξ_1	ξ_2	ξ_3	ξ_4	ξ_5	ξ_6	ξ_7	ξ_8	ξ_9	ξ_{10}	f_{error}
0.4927	0.2839	0.0939	-0.2991	0.2377	-0.1478	-0.0670	0.2377	-0.2613	0.4022	0.0009

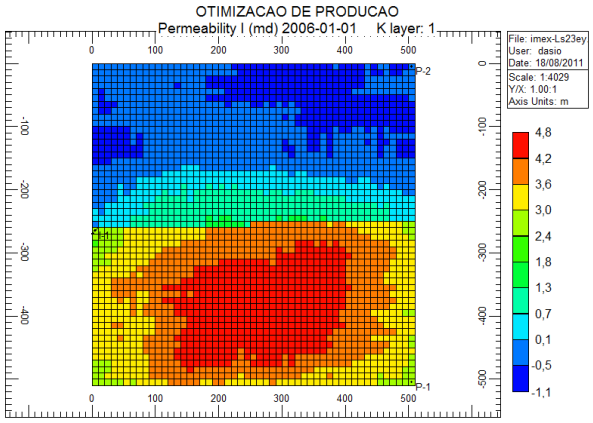

Figure 8. Initial estimate of the permeability field.

Figure 9 ilustrates respectivaly the initial (blue), observed (green) and calculated (red) commulative oil distributions. As before in the linear case, the calculated curve exactly matches the observed case, meaning that the use of ten KL principal components suffice to capture the provided field information. Also, means that a reliable prediction for future field performace can be obtained for the oil field production considering the obtained ξ parameters. This is easily observed in the extrapolation part of the curve, in which the calculated prediction matches the observed production prediction for the extra investigated 6 years (2016 – 2022).

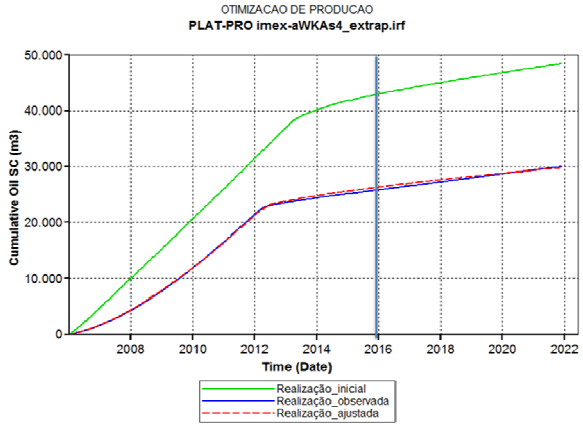

Figure 9. Curves of cumulative oil production obtained in the history matching process.

The permeability Field obtained with the optimum coefficients of the Table 2 is indicated in Figure 10. As can be seen, differently from the linear case, this realization captures the main features of the observed one such as the permeability contrast of two regions as well as the low and high permeability fields zones. Moreover, the total number of numerical simulation for the first case (linear KKLE) was 5676 against only 726 in the nonlinear case.

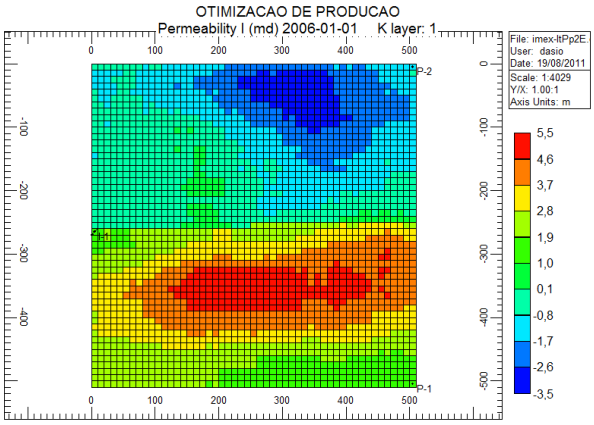

Figure 10. Permeability Field obtained in the history matching process.

9. Conclusions

The history matching in oil reservoirs was here clearly pointed as a complex problem. The challenges in this field are so diverse, involving the mathematical, physical and computational aspects. The main conclusions of the work are presented below:

1. The use of surrogate models is a computationally feasible alternative for solving history matching problems in oil reservoir where the gradients of the problem are not available or will require a high computational cost to be obtained.

2. The SAO methodology used together with KKLE method when applied to history matching problems achieved consistent results. The nonlinear KKLE method could capture complex geological characteristics of the permeability field.

10. References

1. Oliver, D. S., Reynolds, A. C, Liu, N. Theory for Petroleum Reservoir Characterization and History Matching, Cambridge University Press, 2008.
2. Sarma, P., Durlosfky, L. J., Aziz, K., Kernel Principal Component Analysis for Efficient Differentiable Parameterization of Multipoint Geostatistics. Math Geosci 40:3-32, 2008.
3. Forrester, I. J., Sóbester, A., Keane, A. J. Engineering Design via Surrogate Modeling, John Wiley and Sons, 2008.
4. Simpson, T. W, Maurey, T.M , Korte, J.K , Mistree, F. Kriging Models for Global Approximations in simulation-Based Multidisciplinary Design Optimization, AIAA Journal, 39(12), 2233-2241, 2001.
5. Giunta, A. A., Use of Data Sampling, Surrogate Models, and Numerical Optimization in Engineering Design, Proceedings of the 40th AIAA Aerospace Sciences Meeting and Exhibit, Reno, NV, 2002.
6. Powell, M. J. D. "Algorithms for Nonlinear Constraints that use Lagrangian Function", Math. Programming, 14, 224-248, 1978.
7. Alexandrov, N., Dennis Jr., J.E., Lewisand, R.M. and Torezon, V. A Trust Region Framework for Managing the Use of Approximation Models in Optimization, NASA/CR-201745; ICASE Report No. 97-50, 1997.
8. Cmg, Computing Modeling Group LTD. IMEX: User's Guide. Calgary – Canada, 2006.
9. Lira Jr, J. D. Otimização com Modelos Substitutos Considerando Incertezas em Reservatório de Petróleo. Phd Thesis, Civil Engineering, UFPE, Brazil, 2012.
10. Tatang, M. Direct Incorporation of Uncertainty in Chemical and Environmental Enginering Systems. Phd Thesis, MIT, 1995.
11. Huang, S. P., Quek, S. T., Phonn, K. K. Convergence study of the truncated Karhunen–Loeve expansion for simulation of stochastic processes. International Journal for Numerical Methods in Engineering. 52:1029–1043, 2001.
12. Du, Q.,Vance, F., Gunzburger, M. Centroidal Voronoi Tessellations: Applications and Algorithms, Society for Industrial and Applied Mathematics, 1999.
13. Afonso, S. M., Horowitz, B., Lira Junior, J. D., Carmo,, A., Cunha, J., Willmerdorf, R. Comparison of Surrogate Building Techniques for Engineering Problems. XXIX Cilamce, 2008.
14. Keane, A.J., Nair, P.B., Computational Approaches for Aerospace Design: The Pursuit of Excellence, John-Wiley and Sons, 2005.
15. Eldred, M.S., Giunta, A.A. and Collis, S.S. Second-Order Corrections for Surrogate-Based Optimization with Model Hierarchies, Paper AIAA-2004-4457 in Proceedings of the 10th AIAA/ISSMO Multidisciplinary Analysis and Optimization Conference, Albany, NY, 2004
16. Giunta, A. A., Eldred, M. S. Implementation of a Trust Region Model Management Strategy in The Dakota Optimization Toolkit, AIAA-2000-4935, 2000.
17. Eldred, M. S., et al., "Dakota - User´s Manual", 2008.
18. Ghanem, R., "Probabilistic Characterization of Transport in Heterogeneous Media". Computer Methods in Applied Mechanics and Engineering, Elservier, 158 199-220, 1998.
19. Scholkopf, B., Smola, A, Muller K., "Nonlinear Component Analysis as a Kernel Eigenvalue Problem", Technical Report No. 44, Max-Planck Institut für biologische Kybernetik, Arbeitsgruppe Bülthoff, 1996.